Spin Hall Edge Spin Polarization in a Ballistic 2D Electron System

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(Received 16 May 2007; published 7 September 2007)

Universal properties of the spin Hall effect in ballistic 2D electron systems are addressed. The net spin polarization across the edge of the conductor is second order, $\sim\lambda^2$, in spin-orbit coupling constant independent of the form of the boundary potential, with the contributions of normal and evanescent modes each being $\sim\sqrt{\lambda}$ but of opposite signs. This general result is confirmed by the analytical solution for a hard-wall boundary, which also yields the detailed distribution of the local spin polarization. The latter shows fast (Friedel) oscillations with the spin-orbit coupling entering via the period of slow beatings only. Long-wavelength contributions of evanescent and normal modes exactly cancel each other in the spectral distribution of the local spin density.

DOI: 10.1103/PhysRevLett.99.106601 PACS numbers: 72.25.−b, 73.23.−b

Introduction.—Spintronics addresses the interplay of spin and orbital degrees of freedom in various transport, optical, etc., phenomena with the ultimate goal of achieving spin manipulation in nanostructures. A special place in spintronics belongs to the spin Hall effect predicted a long time ago [1], which recently entered the era of experimental observation [2–4]. The spin Hall effect is characterized by a boundary (edge) spin polarization resulting when electric current is flowing through the system. It is customarily classified into "extrinsic" (impurity-driven) [5–8] and "intrinsic" (band-structure induced) [9,10] types. Initially, theories of spin Hall effect addressed such auxiliary quantity as spin current (for a review see Refs. [11]) in infinite systems, but later the emphasis shifted towards direct calculation of spin polarization in confined geometries. For diffusive systems, the search is to complement the coupled spin-density diffusion equations [12,13] with suitable boundary conditions [14–19].

While it is now understood that in 2D systems the spin Hall effect generally occurs with more complicated spin-orbit couplings, any amount of disorder destroys the spin Hall effect in infinite systems with linear coupling [11]. It is, therefore, important to establish whether pure ballistic systems (without disorder) can exhibit nonzero spin Hall polarization. Driven by this motivation, studies of the intrinsic spin Hall effect in ballistic finite-size systems had been initiated, mostly by means of numerical methods [20,21]. It is significant to realize that the edge spin polarization in ballistic systems appears not as a result of electric-field-driven acceleration of electron momenta (and associated with it precession of spins). As is well known, an electric field is absent inside an ideal ballistic conductor connected to reflectionless leads [22]. Spin Hall spin accumulation in ballistic systems is due to the edge precession only. When the populations of left-moving and right-moving states are different, the boundary scattering results in oscillatory (Friedel) edge polarization, which is perpendicular to both the electric current and the normal direction to the boundary. Such polarization was considered numerically in Refs. [23] for a 2D electron gas (2DEG). The case of a 3D hole semiconductor has also been analyzed recently [24]. A possibility of distinguishing edge effects from spin transport has been addressed experimentally in Ref. [25]. Edge spin polarization in parabolic quantum wires has been considered in Ref. [26].

In the present Letter we resolve analytically the boundary problem for a ballistic 2D electron gas with linear spin-orbit coupling [27] and calculate the nonequilibrium edge spin polarization in a wide strip connected to ideal leads with chemical potentials shifted by the applied voltage. We present a general argument that the out-of plane spin polarization integrated over the lateral direction has a *universal* value, *independent* of the particular shape of the confining boundary potential $U(x)$. In the limit of weak spin-orbit coupling, $\lambda \ll v_F$,

$$\int_{-\infty}^{\infty} s_z(x)dx = -\frac{\lambda^2 eV}{12\pi^2 v_F^3}, \tag{1}$$

where eV is the difference of the chemical potentials in the two leads, and v_F is the bulk value of the Fermi velocity (which is the same for both spin-split subbands).

We then illustrate how this result arises from microscopic calculations in a model of a sharp boundary by obtaining the electron Green's functions in a concise analytical form. The obtained spin density is approximated by the expression ($\hbar = 1$),

$$s_z(x) \approx \frac{eV}{2\pi^2 v_F x}\cos(2mv_F x)\sin^2(m\lambda x). \tag{2}$$

It is remarkable that the spin-orbit coupling constant enters via the period of beating only.

Net spin polarization.—Consider a semi-infinite ballistic 2DEG described by the Hamiltonian

$$H = \int d\mathbf{r}\hat{\psi}^\dagger\left[-\frac{\boldsymbol{\partial}^2}{2m} - i\lambda(\hat{\sigma}_x\partial_y - \hat{\sigma}_y\partial_x) + U(x)\right]\hat{\psi}, \tag{3}$$

where potential $U(x)$ ensures boundary confinement (see

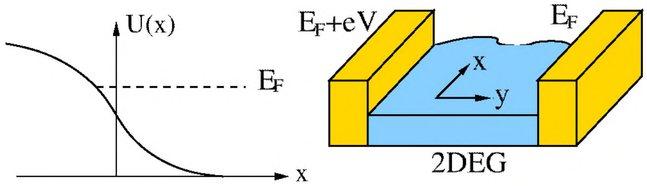

FIG. 1 (color online). Geometry of the system. Ideal leads filled by equilibrium electrons up to the chemical potentials shifted by the applied bias. The edge is formed by a confining potential $U(x)$ vanishing for $x \to \infty$.

Fig. 1). For the sake of simplicity we present derivation for the case of Rashba spin-orbit interaction, though calculations for the Dresselhaus coupling [28] are completely analogous [29]. The system is attached to two ideal reflectionless leads injecting equilibrium electrons into 2DEG. The chemical potentials of the leads are shifted by the applied voltage, eV.

Since k_y is an integral of motion (in the case of reflectionless leads), it is convenient to use the Fourier representation along the y axis for the electron operators, $\hat{\psi}(\mathbf{r}) = \sum_{k_y} \hat{c}_{k_y}(x) e^{ik_y y}$. One can now derive the equation of motion for the expectation value of the electron spin operator, $\mathbf{s}(k_y, x) = \frac{1}{2}\langle \hat{c}^\dagger_{k_y}(x) \hat{\boldsymbol{\sigma}} \hat{c}_{k_y}(x)\rangle$, which can be readily written in the form,

$$\partial_t s_y(k_y, x) = -\partial_x J^y_x(k_y, x) - 2\lambda k_y s_z(k_y, x). \tag{4}$$

Here J^y_x stands for the conventional operator of spin current, i.e.,

$$J^y_x(k_y, x) = \frac{i}{4m}\langle \nabla_x \hat{c}^\dagger_{k_y} \hat{\sigma}_y \hat{c}_{k_y} - \hat{c}^\dagger_{k_y} \hat{\sigma}_y \nabla_x \hat{c}_{k_y}\rangle - \frac{\lambda}{2}\langle \hat{c}^\dagger_{k_y} \hat{c}_{k_y}\rangle.$$

In a steady state the left-hand side of Eq. (4) vanishes. Integrating Eq. (4) over the x direction, we obtain for the net spin polarization,

$$\int_{-\infty}^{\infty} s_z(x) dx = -\frac{1}{2\lambda} \sum_{k_y} \frac{1}{k_y} J^y_x(k_y, \infty). \tag{5}$$

It is straightforward to calculate the value of the (k_y-resolved) spin current $J^y_x(k_y, \infty)$ inside the bulk of a 2D system:

$$J^y_x(k_y, \infty) = -\frac{1}{2} \sum_{\beta=\pm 1} \sum_{k_x} \left(\lambda + \frac{\beta k_x^2}{mk}\right) n_\beta(k_x, k_y), \tag{6}$$

where $n_\beta(k_x, k_y)$ stands for the population of different momentum states in the subband β. Only "uncompensated" states contribute to the nonequilibrium spin polarization given by Eqs. (5) and (6); these states describe electrons that originate in the left lead ($k_y > 0$) and belong to the energy interval near the Fermi energy, $E_F < k^2/2m + \beta k \lambda < E_F + eV$. The integral (5) diverges logarithmically at $k_y \to 0$. Assuming the same infrared cutoff in both subbands, $\tilde{k}$, we observe that the diverging $\ln \tilde{k}$ contributions in the two subbands cancel each other, yielding in the linear (in V) response,

$$\int_{-\infty}^{\infty} s_z dx = \frac{eV}{2\lambda(2\pi)^2}\left(\frac{2\lambda}{v_F} - \ln\frac{v_F + \lambda}{v_F - \lambda}\right), \tag{7}$$

where $v_F = \sqrt{2E_F/m + \lambda^2}$ is the Fermi velocity. Expanding this *general* result to the lowest nonvanishing order in λ/v_F we recover the net boundary polarization, Eq. (1).

Evanescent modes.—The reflection at the boundary mixes the two bulk subbands. Those states that belong to the domain, $k^+ < k_y < k^-$, where $k^\pm = m(v_F \mp \lambda)$, refered to as evanescent states [23], are characterized by exponentially decaying contribution from the upper (+) subband. Repeating the calculations leading to Eq. (7), but now for the evanescent domain only, we obtain

$$\int_{-\infty}^{\infty} s_z^{\mathrm{ev}} dx = \frac{eV}{2\lambda(2\pi)^2}\left(2\sqrt{\frac{\lambda}{v_F}} - \ln\frac{1 + \sqrt{\lambda/v_F}}{1 - \sqrt{\lambda/v_F}}\right). \tag{8}$$

Remarkably, the net evanescent contribution is $\sim\sqrt{\lambda}$ and is *largely* canceled by the contribution from the normal domain $k_y < k^+$, yielding Eq. (1) which is quadratic in λ. This cancellation occurs for local spin density as well; see Eq. (2).

Electron Green's function.—Microscopic calculation of the local spin polarization can be most simply performed with the help of the electron Green's functions,

$$\mathbf{s}(x) = i\frac{eV}{4\pi} \mathrm{Tr} \int_0^{k^-} \frac{dk_y}{2\pi} [G^R_{k_y E_F}(x, x) - G^A_{k_y E_F}(x, x)] \hat{\boldsymbol{\sigma}}, \tag{9}$$

where $G^R_{k_y E}(x, x')$ is the retarded Green's function. Its advanced counterpart satisfies the condition $G^A_{k_y E}(x, x') = G^{R\dagger}_{k_y E}(x', x)$. The summation over energy in Eq. (9) is performed over the narrow strip of width eV, similar to Eqs. (6) and (7).

To illustrate how spin polarization arises from the solution of the Schrödinger equation, let us solve the problem of a hard-wall boundary: $U(x) = 0$, for $x > 0$ and $U(x) = \infty$ for $x < 0$. The case of a smooth boundary where electrons adiabatically follow semiclassical trajectories for spin-split subbands [30] will be considered separately [31].

For a plane wave, $\sim e^{ik_y y}$, the equation for the Green's functions for ($x, x' > 0$) is

$$\left[\frac{\partial_x^2}{2m} - \lambda(i\hat{\sigma}_y \partial_x + \hat{\sigma}_x k_y) + E'\right] \hat{G}(x, x') = -\delta(x - x'), \tag{10}$$

where the subscripts k_y and E are omitted for simplicity, and $E' = E - k_y^2/2m$. The boundary condition for the impenetrable wall is $G(x, 0) = G(0, x') = 0$. We solve the problem by first noting that the following function $\hat{\mathcal{L}}(x)$ satisfies both the homogeneous Eq. (10) and the boundary condition $\hat{\mathcal{L}}(0) = 0$,

$$\hat{\mathcal{L}}(x) = \frac{1}{i\sum_\beta k^\beta} \sum_\beta \frac{k^\beta}{k_x^\beta} (e^{ik_x^\beta x} \hat{B}_\beta - e^{-ik_x^\beta x} \hat{B}^*_\beta), \tag{11}$$

here $*$ stands for the simple complex (not Hermitian) conjugate; the sum is taken over both subbands, with the projection matrix for the subband β defined as

$$B_\beta = \frac{1}{2}\left(1 + \beta\frac{k_y}{k^\beta}\hat{\sigma}_x - \beta\frac{k_x^\beta}{k^\beta}\hat{\sigma}_y\right), \tag{12}$$

where the absolute value of the electron momentum k^β is defined above Eq. (8) and its x component is $k_x^\beta = \sqrt{(k^\beta)^2 - k_y^2}$. Here we concentrate on the normal modes, where both $k_x^\pm$ are real; rather simple modifications for the evanescent domain (where k_x^+ is imaginary) are outlined below.

Using the function (11) we can readily construct the solution for the inhomogeneous Eq. (10) which satisfies the boundary condition $G(0, x') = 0$,

$$\hat{G}(x, x') = -2m[\hat{\mathcal{L}}(x)\hat{A}(x') + \Theta(x - x')\hat{\mathcal{L}}(x - x')], \tag{13}$$

where $\hat{A}(x')$ is a yet unknown matrix. Since the Green's function has to obey both the boundary condition $G(x, 0) = 0$ and the equation conjugated to Eq. (10), the matrix $\hat{A}(x')$ must be a homogeneous solution satisfying the condition $\hat{A}(0) = -1$. This determines it up to some constant matrix $\hat{C}$ different for the retarded and advanced Green's functions, $\hat{A}(x') = \hat{C}\hat{\mathcal{L}}^\dagger(x') - \partial_{x'}\hat{\mathcal{L}}^\dagger(x')$,

$$\hat{G}_{R,A}(x, x') = -2m[\hat{\mathcal{L}}(x)\hat{C}_{R,A}\hat{\mathcal{L}}^\dagger(x') - \hat{\mathcal{L}}(x)\partial_{x'}\hat{\mathcal{L}}^\dagger(x') + \Theta(x - x')\hat{\mathcal{L}}(x - x')]. \tag{14}$$

The constant $\hat{C}_R$ ($\hat{C}_A$) is most simply determined from the condition that the retarded (advanced) Green's function does not contain the waves $\sim e^{-ik_x^\beta x}$ ($e^{ik_x^\beta x}$) propagating to (from) the boundary in the region $x > x'$. The calculations are straightforward but rather tedious. As a result one obtains,

$$\hat{C}_{R,A} = \mp\frac{i}{2}(k_x^+ + k_x^-) \mp \frac{i}{2k_y}(k^- k_x^+ - k^+ k_x^-)\hat{\sigma}_x - im\lambda\hat{\sigma}_y + \frac{1}{2k_y}(k^+k^- - k_y^2 - k_x^+k_x^-)\hat{\sigma}_z, \tag{15}$$

where the upper (lower) sign corresponds to $\hat{C}_R(\hat{C}_A)$.

Spin polarization.—Making use of the derived Green's function we can now calculate the local spin polarization (9). With the help of Eqs. (14) and (15) we obtain

$$\begin{aligned} s_z(x) = \frac{eV}{2(2\pi)^2 m v_F^2}\Bigg(&\int_0^{k^+}\frac{dk_y}{k_y}(k^+k^- + k_y^2 - k_x^+k_x^-) \\ &\times\{\sin(2k_x^+x) + \sin(2k_x^-x) - 2\sin[(k_x^+ + k_x^-)x]\} \\ &- \int_{k^+}^{k^-}\frac{dk_y}{k_y}\{k_x^-\kappa(e^{ik_x^-x} - e^{-\kappa x})^2 \\ &- 2(k^+k^- + k_y^2 + ik_x^-\kappa)\sin(k_x^-x) \\ &\times[\cos(k_x^-x) - e^{-\kappa x}]\}\Bigg). \end{aligned} \tag{16}$$

The first line here is the contribution of the normal modes while the second line comes from the evanescent modes, where $\kappa = \sqrt{k_y^2 - (k^+)^2}$ [32]. By calculating the integral over x it is straightforward to verify that the net contribution of the evanescent modes satisfies Eq. (8), being $\sim\sqrt{\lambda}$. This is mostly cancelled by the contribution from the normal modes. The total net contribution of both the normal and evanescent states yields Eq. (7) in agreement with our general argument based on the equation of motion for spin operators, Eq. (4). The behavior of the local spin density (16) is shown in Fig. 2. In the most relevant limit, $\lambda \ll v_F$, Eq. (16) can be simplified to

$$\begin{aligned} s_z(x) = &-eV\frac{m\lambda}{\pi^2 v_F}\int_0^1 dy\sin(x\sqrt{1-y}/\xi)e^{-x\sqrt{y}/\xi} \\ &+ \frac{eV}{16\pi^2 m v_F^2 x^2}\sum_\beta[\sin(2k^\beta x) - 2k^\beta x\cos(2k^\beta x)] \\ &- \frac{eV(k^+)^2}{4\pi^2 m v_F^2}\int_0^1 dy\sin[xk^+(\sqrt{y} + \sqrt{\delta + y})], \end{aligned} \tag{17}$$

where $\xi^{-1} = 2m\sqrt{\lambda v_F}$, and $\delta = (k^-/k^+)^2 - 1$. Integrating this expression over x, we recover the net spin polarization (1).

Spectral distribution of spin density.—It is instructive to present the results in terms of the Fourier transform of the spin density, $s_z(q) = 2\int_0^\infty dx s_z(x)\sin qx$. From Eq. (16) we find $s_z(q)$ in a form of piecewise continuous algebraic function defined in four domains. The surprising feature of the spectral distribution revealed by this calculation is its vanishing, $s_z(q) = 0$, in the whole long-wavelength domain, $0 < q < 2k^+$. In particular, this shows the exact cancellation between normal and evanescent modes. For larger values of q we obtain to the leading order in λ,

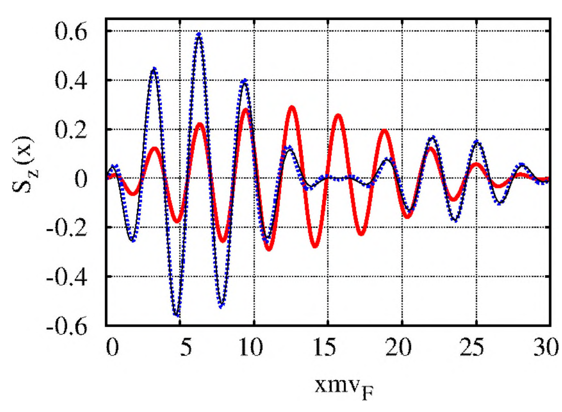

FIG. 2 (color online). Dependence of the local spin polarization (16), in units of $eVm/8\pi^2$, on the distance to the boundary for different values of spin-orbit coupling constant. Solid (red) line: $\lambda/v_F = 0.1$, dotted (blue) line: $\lambda/v_F = 0.2$, solid (black) line utilizes the approximate formula (2) for $\lambda/v_F = 0.2$. The plot of Eq. (17) is indistinguishable from the exact Eq. (16) on this scale.

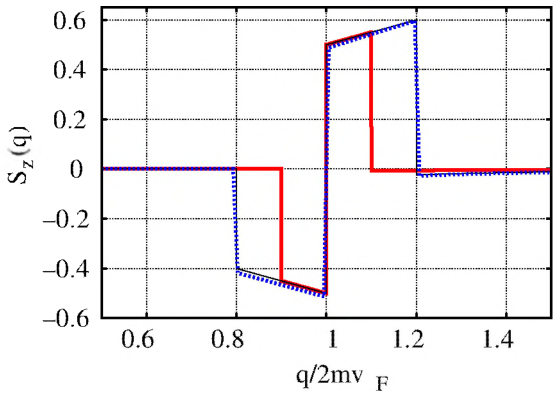

FIG. 3 (color online). Spectral distribution (17) of spin density in units of $eV/4\pi v_F$ for different values of spin-orbit coupling constant. Solid (red) line: $\lambda/v_F = 0.1$. Solid (black) line: $\lambda/v_F = 0.2$. Dotted (blue) line shows Fourier transform of the exact Eq. (16) for $\lambda/v_F = 0.2$.

$$s_z(q) = \frac{eVq}{16\pi m v_F^2} \times \begin{cases} -1, & 2k^+ < q < 2mv_F, \\ 1, & 2mv_F < q < 2k^-, \\ -2/(q\xi)^4, & 2k^- < q. \end{cases} \tag{18}$$

The plot of the spectral distribution is illustrated in Fig. 3. Remarkably, the net spin polarization [given by $\pi^{-1}\int dq s_z(q)/q$] comes from the large-q tail ($\propto q^{-3}$) in the spectral density $s_z(q)$.

Conclusion.—In this Letter we solved analytically a problem of mesoscopic spin Hall effect in a confined 2D electron system. We presented general arguments why the net spin polarization in a ballistic spin Hall effect is *independent* of the boundary potential and confirmed the result by a straightforward calculation for the hard-wall boundary, for which the analytical solution was obtained. The spectral distribution of spin density consists of two narrow peaks of opposite sign whose heights are virtually independent of the small spin-orbit coupling constant. Surprisingly, long-wavelength contributions from evanescent and normal modes *exactly* cancel each other. Understanding the level of universality of this cancellation for arbitrary boundary potentials remains a challenging problem.

We acknowledge fruitful discussions with A. Andreev, B. Halperin, M. Raikh, and O. Starykh. The work was supported by DOE, Grant No. DE-FG02-06ER46313 and by SFB TR 12.

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